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Design of bounded feedback controls for linear dynamical systems by using common Lyapunov functions

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Abstract For a linear dynamical system, we address the problem of devising a bounded feedback control, which brings the system to the origin in finite time. The construction is based on the notion of a common Lyapunov function. It is shown that the constructed control remains effective in the presence of small perturbations. [doi:10.1063/2.1101301]

Keywords linear control system, feedback control, bounded control, common Lyapunov function

Consider a linear autonomous dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \ \boldsymbol{x} \in V = \mathbf{R}^n, \ \boldsymbol{u} \in U = \mathbf{R}^m, \tag{1}$$

such that the Kalman controllability condition is met. We want to build a bounded feedback control $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$, which brings an arbitrary state $\boldsymbol{x}_0$ to the origin in finite time, provided that $|\boldsymbol{x}_0|$ is small enough. In other words, the corresponding phase curves of equation $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{x})$ with the initial conditions $\boldsymbol{x}(0) = \boldsymbol{x}_0$ gets to 0 in finite time. Note that, given a bound $|\boldsymbol{u}| \le C$ on control, it is generally impossible to steer any given initial state into the origin.

The problem of feedback control design has been studied, in particular, by V. Korobov,[1] and his paper is a starting point for ours, though our arguments can be hardly put into a direct comparison with that of Ref. [1]. In principle, to get to the zero one can fix the bound $|\boldsymbol{u}| \le C$, and use the minimum time control $\boldsymbol{u}_{\min}$.[2,3] The obvious drawback of this approach consists in the great difficulties of implementation: the amount of computations required is prohibitive for a numerical simulation. Therefore we need the feedback control to be devised in such a way as to be easily implementable (constructive). One can see *a posteriori* that our control algorithm does not require much memory or computational power. To implement it one needs just basic operations of linear algebra plus finding the only root of a scalar monotone function of one variable. Our control is more smooth than the minimum–time one: its only singular point is zero, while the singular locus of optimal control is a singular hypersurface. Moreover, the time $\tau(\boldsymbol{x})$ required for our control to bring a given state $\boldsymbol{x}$ to 0 is not much greater than the minimal one $\tau_{\min}(\boldsymbol{x})$: the ratio $\tau(\boldsymbol{x})/\tau_{\min}(\boldsymbol{x})$ is bounded as $\boldsymbol{x}$ runs over a neighborhood of zero. In our terminology, the feedback control $\boldsymbol{u}$ is locally equivalent to the minimum-time control $\boldsymbol{u}_{\min}$.

First, we simplify our control system (1). Note that the feedback control problem does not change essentially under transformation $\boldsymbol{A} \mapsto \boldsymbol{A} + \boldsymbol{B}\boldsymbol{C}$, $\boldsymbol{u} \mapsto \boldsymbol{u} - \boldsymbol{C}\boldsymbol{x}$, of (1) corresponding to an extra linear feedback. Moreover, for any invertible matrix $\boldsymbol{D}$ the gauge transformation $\boldsymbol{A} \mapsto \boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}$, $\boldsymbol{B} \mapsto \boldsymbol{D}^{-1}\boldsymbol{B}$, $\boldsymbol{u} \mapsto \boldsymbol{u}$ does not affect the problem. By using these transformations one can bring system (1) to the canonical Brunovsky form [4–6] — a set of independent subsystems of the form $z^{(k)} = u; z, u \in \mathbf{R}^1$. Now it suffices to bring each subsystem $z^{(k)} = u$ to zero by a bounded feedback control.

Thus, the general problem (1) reduces to the case

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}, \ \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{2}$$

We introduce a scalar function $T = T(\boldsymbol{x})$ which is specified below. System (1), (2) is related to a distinguished function matrix

$$\delta(T) = \operatorname{diag}(T^{-n}, T^{-n+1}, \ldots, T^{-1})$$

such that

$$\delta\boldsymbol{A}\delta^{-1} = T^{-1}\boldsymbol{A}, \ \delta\boldsymbol{B} = T^{-1}\boldsymbol{B}, \ \frac{\mathrm{d}}{\mathrm{d}T}\delta = T^{-1}\boldsymbol{M}\delta, \tag{3}$$

where $\boldsymbol{M} = -\operatorname{diag}(n, n-1, \ldots, 1)$. This implies immediately that for $\boldsymbol{y} = \delta\boldsymbol{x}$ we have

$$\dot{\boldsymbol{y}} = T^{-1}\left(\boldsymbol{A}\boldsymbol{y} + \boldsymbol{B}\boldsymbol{u} + \dot{T}\boldsymbol{M}\boldsymbol{y}\right). \tag{4}$$

Here we present the main novelty of the paper: a construction of a common Lyapunov function for two specific stable matrices. Our feedback controls are based on the existence of this function.

In notations (2) consider stable matrices $\widetilde{\boldsymbol{A}}$ of the form

$$\widetilde{\boldsymbol{A}} = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{C}, \tag{5}$$

where the row-vector $\boldsymbol{C} = (c_1, \ldots, c_n)$ is regarded as a $1 \times n$ matrix. In other words,

$$\widetilde{\boldsymbol{A}} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ c_1 & c_2 & c_3 & \ldots & c_n \end{pmatrix},$$

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and the polynomial $f(\boldsymbol{x}) = x^n - \sum\limits_{i=1}^{n} c_i x^{i-1}$ is stable, i.e. all its roots have a negative real part.

Theorem 1 One can choose the vector $\boldsymbol{C}$ in such a way that the stable matrices $\widetilde{\boldsymbol{A}}$ and $\boldsymbol{M}$ possess a common quadratic Lyapunov function: There exist symmetric positive definite matrices $\boldsymbol{Q}, \boldsymbol{P}$, and $\boldsymbol{R}$ such that

$$\boldsymbol{Q}\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{A}}^*\boldsymbol{Q} = -\boldsymbol{P}, \quad \boldsymbol{QM} + \boldsymbol{MQ} = -\boldsymbol{R} \tag{6}$$

Remark. The vector $\boldsymbol{C}$ can be defined as follows Put

$$f_\lambda(\boldsymbol{x}) = \prod_{k=1}^{n}(\boldsymbol{x} + \mathrm{e}^{\lambda k}) = x^n - \sum_{i=1}^{n} c_i x^{i-1}.$$

Then $\boldsymbol{C}$ fits the theorem, provided that $\lambda > 0$ is large enough.

Now we can define a bounded feedback control u which brings the system (1), (2) to zero in finite time. Put $\boldsymbol{y} = \delta(T)\boldsymbol{x}$, $\boldsymbol{u}(\boldsymbol{x}) = (\boldsymbol{C}, \boldsymbol{y})$, where (the row-vector) $\boldsymbol{C}$ is chosen in Theorem 1. We define the function T by $T(0) = 0$ and

$$(\boldsymbol{Qy}, \boldsymbol{y}) = 1, \text{ where } \boldsymbol{y} = \delta(T)\boldsymbol{x} \text{ if } \boldsymbol{x} \neq 0. \tag{7}$$

The definition is correct, since for a fixed $\boldsymbol{x} \neq 0$ the analytic function $\phi(T) = (\boldsymbol{Q}\delta(T)\boldsymbol{x}, \delta(T)\boldsymbol{x})$ decreases as T increases, and tends to infinity as $T \to 0$, and to zero as $T \to \infty$. Indeed, by virtue of Theorem 1

$$\frac{\mathrm{d}}{\mathrm{d}T}\phi(T) = T^{-1}\left((\boldsymbol{QM} + \boldsymbol{MQ})\boldsymbol{y}, \boldsymbol{y}\right) < 0 \tag{8}$$

Moreover, T depends on $\boldsymbol{x}$ analytically if $\boldsymbol{x} \neq 0$, and the condition (7) guarantee the boundedness of $\boldsymbol{u}(\boldsymbol{x}) = (\boldsymbol{C}, \boldsymbol{y}(\boldsymbol{x}))$.

Now it follows from Eqs. (7) and (4) that

$$\left(\boldsymbol{y}, (\boldsymbol{Q}\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{A}}^*\boldsymbol{Q})\boldsymbol{y} + \dot{T}(\boldsymbol{QM} + \boldsymbol{MQ})\boldsymbol{y}\right) = 0,$$

or

$$\dot{T} = -\frac{(\boldsymbol{y}, (\boldsymbol{Q}\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{A}}^*\boldsymbol{Q})\boldsymbol{y})}{(\boldsymbol{y}, (\boldsymbol{QM} + \boldsymbol{MQ})\boldsymbol{y})}.$$

In view of the Lyapunov equations (6)

$$\dot{T} \leq -c,$$

where $c = c(\boldsymbol{Q})$ is a positive constant. This implies that up to the zero T decreases with a speed separated from 0. Therefore, the motion ends in the zero in finite time $\tau(\boldsymbol{x})$ which can be estimated as $\tau(\boldsymbol{x}) = O(T(\boldsymbol{x}))$. In its turn, $T(\boldsymbol{x})$ can be estimated as $O(\tau_{\min}(\boldsymbol{x}))$ so that the time required for getting into zero is of the same order of magnitude as the minimal one. The result of this section can be stated as follows:

Theorem 2 Suppose $\boldsymbol{Q}(\boldsymbol{x}) = (\boldsymbol{Qx}, \boldsymbol{x})$ is a common quadratic Lyapunov function for two stable matrices $\widetilde{\boldsymbol{A}} = \boldsymbol{A} + \boldsymbol{BC}$ and $\boldsymbol{M}$. Then, condition (7) defines a bounded feedback control $u(\boldsymbol{x}) = (\boldsymbol{C}, \boldsymbol{y}(\boldsymbol{x})) = (\boldsymbol{C}, \delta(T(\boldsymbol{x}))\boldsymbol{x})$ bringing any state vector of system (1), (2) to zero in finite time. This time has the same order of magnitude as the minimal one.

Remark. Note that the proposed control is global: it is bounded in the whole phase space and brings any initial state of system (1), (2) to zero in finite time. It also remains effective for the system

$$z^{(n)} = u + v$$

under small perturbation $\boldsymbol{v}$.

One can generalize the above first method of control as follows: We again put $\boldsymbol{y} = \delta(T)\boldsymbol{x}$, $\boldsymbol{u}(\boldsymbol{x}) = (\boldsymbol{C}, \boldsymbol{y})$, but define the function T by condition

$$T^{-2\beta}\left(\boldsymbol{Qy}, \boldsymbol{y}\right) = 1, \tag{9}$$

where $\beta \geq 0$ is a new parameter. Introduction of the new parameter does not spoil our previous arguments essentially. The function $\phi_\beta(T) = T^{-2\beta}$ $(\boldsymbol{Q}\delta(T)\boldsymbol{x}, \delta(T)\boldsymbol{x})$ tends to infinity as $T \to 0$, and to zero as $T \to \infty$. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}T}\phi_\beta(T) = T^{-1-2\beta}\left((\boldsymbol{QM}_\beta + \boldsymbol{M}_\beta\boldsymbol{Q})\boldsymbol{y}, \boldsymbol{y}\right), \tag{10}$$

where $\boldsymbol{M}_\beta = \boldsymbol{M} - \beta\boldsymbol{I}$. If the matrix $\boldsymbol{Q}$ defines a quadratic Lyapunov function for the stable matrix $\boldsymbol{M}_\beta$, then we see from (10) that $\phi_\beta(T)$ decreases as T increases. This allows us to define the function $T = T(\boldsymbol{x})$. Similarly to our arguments in the previous section it follows from Eqs. (9) and (4) that

$$\dot{T} = -\frac{(\boldsymbol{y}, (\boldsymbol{Q}\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{A}}^*\boldsymbol{Q})\boldsymbol{y})}{(\boldsymbol{y}, (\boldsymbol{QM}_\beta + \boldsymbol{M}_\beta\boldsymbol{Q})\boldsymbol{y})}. \tag{11}$$

If the matrix $\boldsymbol{Q}$ defines a common quadratic Lyapunov function for two stable matrices $\widetilde{\boldsymbol{A}} = \boldsymbol{A} + \boldsymbol{BC}$ and $\boldsymbol{M}_\beta = \boldsymbol{M} - \beta I$ then the above arguments prove that the controlled motion ends in the zero in finite time $\tau(\boldsymbol{x}) = O(T(\boldsymbol{x}))$.

The result of this section can be stated as follows:

Theorem 3 Suppose $\boldsymbol{Q}(\boldsymbol{x}) = (\boldsymbol{Qx}, \boldsymbol{x})$ is a common quadratic Lyapunov function for two stable matrices $\widetilde{\boldsymbol{A}} = \boldsymbol{A} + \boldsymbol{BC}$ and $\boldsymbol{M}_\beta = \boldsymbol{M} - \beta\boldsymbol{I}$. Then, condition (9) defines a bounded feedback control $\boldsymbol{u}(\boldsymbol{x}) = (\boldsymbol{C}, \boldsymbol{y}(x)) = \boldsymbol{C}\delta(T(\boldsymbol{x}))\boldsymbol{x}$ bringing any state vector of the system (1),(2) to zero in finite time.

Remark. Note that Theorem 2 is based on a rather deep Theorem 1. On the contrary, conditions of Theorem 3 can be easily verified in many cases, e.g. if β is large, without appealing to any deep result. On the other hand, the time for getting to zero needed by Theorem 3 can be much greater than that in Theorem 2.

The second method of control has an advantage in that it still works under smooth perturbations

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{B}u,\ \ \boldsymbol{f}(\boldsymbol{x}) = O(|\boldsymbol{x}|^2) \tag{12}$$

of the control system.

Theorem 4 Suppose $\boldsymbol{Q}(\boldsymbol{x}) = (\boldsymbol{Q}\boldsymbol{x}, \boldsymbol{x})$ is a common quadratic Lyapunov function for two stable matrices $\widetilde{\boldsymbol{A}} = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{C}$ and $\boldsymbol{M}_\beta = \boldsymbol{M} - \beta\boldsymbol{I}$, and $\beta > n-3$. Then, condition (9) defines a bounded feedback control $u(\boldsymbol{x}) = (\boldsymbol{C}, \boldsymbol{y}(\boldsymbol{x})) = (\boldsymbol{C}, \delta(T(\boldsymbol{x}))\boldsymbol{x})$ bringing any state vector close to zero of the system (12), (2) to zero in finite time.

Remark. Thus, the second approach is locally applicable to a nonlinear control system

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{B}u,\ \ \boldsymbol{F}(\boldsymbol{x}) \in \boldsymbol{C}^2$$

which can be represented in the form (12) in the vicinity of an equilibrium state.

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